MTH 295
Fall 2019
Homework 9
Due Thursday, 11/14 Name: Key

1) Systems are analogous to linear equations in many ways (surprise, higher order equations can be reduced to systems).

a) Show that if x_1 and x_2 are each solutions to the matrix equation $Ax=0$ then $c_1x_1+c_2x_2$ is also a solution for any scalars c_1, c_2.

If $AX_1 = AX_2 = 0$ then

$$\begin{aligned} A(c_1X_1+c_2X_2) &= A(c_1X_1)+A(c_2X_2) \\ &= c_1AX_1+c_2AX_2 \\ &= 0+0 \\ &= 0 \end{aligned}$$

and $c_1X_1+c_2X_2$ is a solution of $AX=0$.

b) Show that if x_1 is a solution of $Ax=b$ and x_2 is a solution of $Ax=0$ then x_1+cx_2 is a solution of $Ax=b$ for any scalar c.

If $AX_1=b$ and $AX_2=0$ then

$$\begin{aligned} A(X_1+cX_2) &= AX_1+A(cX_2) \\ &= AX_1+cAX_2 \\ &= b+0 \\ &= b \end{aligned}$$

and X_1+cX_2 is a solution of $AX=b$,

c) Show that if x_1 is a solution of $Ax=b_1$ and x_2 is a solution of $Ax=b_2$ then x_1+x_2 is a solution of $Ax=b_1+b_2$.

If $AX_1=b_1$ and $AX_2=b_2$ then

$$\begin{aligned} A(X_1+X_2) &= AX_1+AX_2 \\ &= b_1+b_2 \end{aligned}$$

and X_1+X_2 is a solution of $AX=b_1+b_2$

2) Solve the IVP $x' = 9x + 5y, y' = -6x - 2y, x(0) = 1, y(0) = 0$ using the matrix method. Please express your solution in the form $x = x(t), y = y(t)$.

In matrix form -

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 9 & 5 \\ -6 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

If $\begin{pmatrix} x \\ y \end{pmatrix} = e^{kt}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ then

$$k\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 9 & 5 \\ -6 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\left\{\begin{pmatrix} 9 & 5 \\ -6 & -2 \end{pmatrix} - k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} 9-k & 5 \\ -6 & -2-k \end{vmatrix} = (9-k)(-2-k)+30$$

$$= k^2 - 7k + 12 = 0$$

$$(k-3)(k-4) = 0$$

$$k = 3, 4$$

If $k=3$ then

$$\begin{pmatrix} 9-3 & 5 \\ -6 & -2-3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

so $6c_1 + 5c_2 = 0$

and $c_2 = -\frac{6}{5}c_1$ and $\begin{pmatrix} 5c_1 \\ -6c_1 \end{pmatrix}$ is an eigenvector with eigenvalue 3

If $k=4$ then $\begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$ so $c_1 = -c_2$

and $\begin{pmatrix} -c_2 \\ c_2 \end{pmatrix}$ is an eigenvector with eigenvalue 4.

then $\begin{pmatrix} x \\ y \end{pmatrix} = e^{3t}\begin{pmatrix} 5c_1 \\ -6c_1 \end{pmatrix} + e^{4t}\begin{pmatrix} -c_2 \\ c_2 \end{pmatrix}$

or $x = 5c_1e^{3t} - c_2e^{4t}$

$y = -6c_1e^{3t} + c_2e^{4t}$

Apply conditions - If $x(0)=1$ and $y(0)=0$ then

$$5c_1 - c_2 = 1$$
$$-6c_1 + c_2 = 0$$

so $-c_1 = 1$

$c_1 = -1$

$c_2 = -6$

So
$$\boxed{\begin{aligned} x &= -5e^{3t} + 6e^{4t} \\ y &= 6e^{3t} - 6e^{4t} \end{aligned}}$$

3) Use the eigenvalue method to find the particular solution of the IVP $x' = x - 2y$, $y' = 2x + y$, $x(0) = 0$, $y(0) = 4$. Express your solution in the form $x = x(t)$, $y = y(t)$.

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{if } \begin{pmatrix} x \\ y \end{pmatrix} = e^{kt}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \text{ then}$$

$$k\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} 1-k & -2 \\ 2 & 1-k \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $\begin{vmatrix} 1-k & -2 \\ 2 & 1-k \end{vmatrix} = k^2 - 2k + 5 = 0$

so $k = \dfrac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$

if $k = 1+2i$ then $\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$ and $-2ic_1 - 2c_2 = 0$
$c_2 = -ic_1$

so $\begin{pmatrix} c_1 \\ -ic_1 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -ic_1 \end{pmatrix}$ or $\begin{pmatrix} c_1 \\ 0 \end{pmatrix} - i\begin{pmatrix} 0 \\ c_1 \end{pmatrix}$ is an eigenvector

so $\begin{pmatrix} x \\ y \end{pmatrix} = e^t\{\cos 2t + i\sin 2t\}\left\{\begin{pmatrix} c_1 \\ 0 \end{pmatrix} - i\begin{pmatrix} 0 \\ c_1 \end{pmatrix}\right\}$

$= e^t\left\{\cos 2t\begin{pmatrix} c_1 \\ 0 \end{pmatrix} + \sin 2t\begin{pmatrix} 0 \\ c_1 \end{pmatrix}\right\} + e^t\left\{-i\cos 2t\begin{pmatrix} 0 \\ c_1 \end{pmatrix} + i\sin 2t\begin{pmatrix} c_1 \\ 0 \end{pmatrix}\right\}$

or $x = e^t(c_1\cos 2t + c_2\sin 2t)$
$y = e^t(c_1\sin 2t - c_2\cos 2t)$ where I let $ic_1 = c_2$

apply conditions –

$x(0) = c_1 = 0$

$y(0) = -c_2 = 4 \Rightarrow c_2 = -4$

so
$$\boxed{\begin{aligned} x &= -4e^t\sin 2t \\ y &= 4e^t\cos 2t \end{aligned}}$$

you should check for errors
clearly, $x(0) = 0$ ✓
$y(0) = 4$ ✓

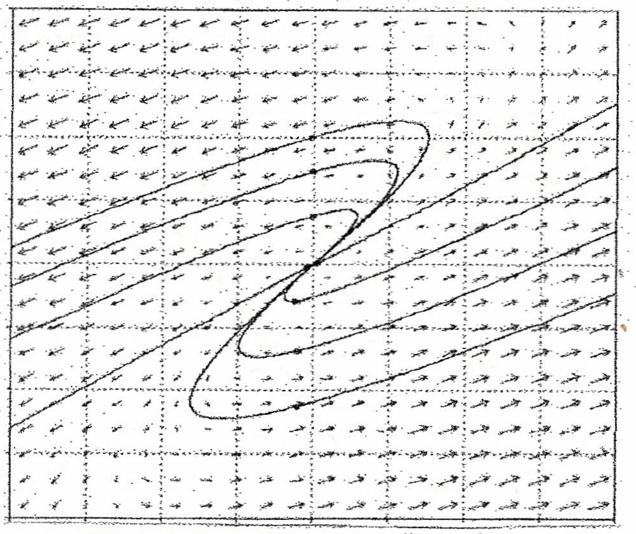
$x' = -4e^t\sin 2t - 8e^t\cos 2t$
$x - 2y = -4e^t\sin 2t - 8e^t\cos 2t$ ✓

$y' = 4e^t\cos 2t - 8e^t\sin 2t$
$2x + y = -8e^t\sin 2t + 4e^t\cos 2t$ ✓

so x & y satisfy the IVP.

4) Label each of the following planar phase diagrams according to the following:

a) Two real positive eigenvalues, possibly repeated.
b) Two real negative eigenvalues, possibly repeated.
c) Two real eigenvalues, one positive, one negative.
d) Two complex eigenvalues with positive real part.
e) Two complex eigenvalues with negative real part.
f) Two purely imaginary eigenvalues.



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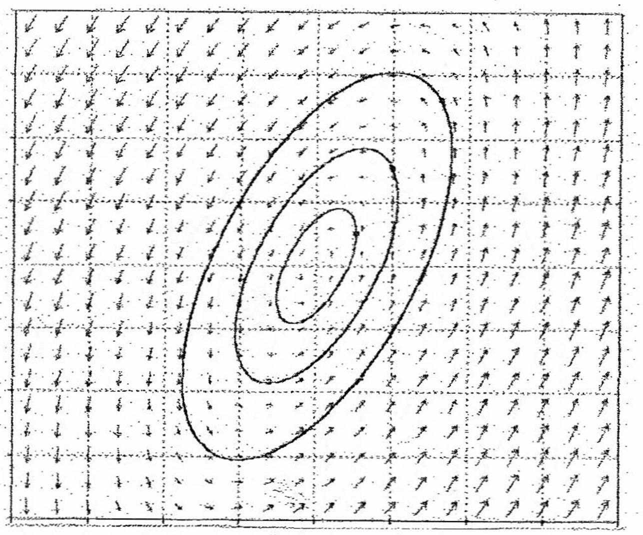

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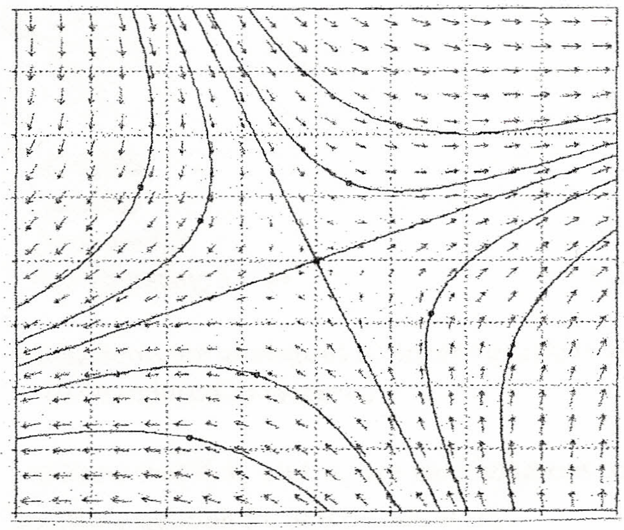

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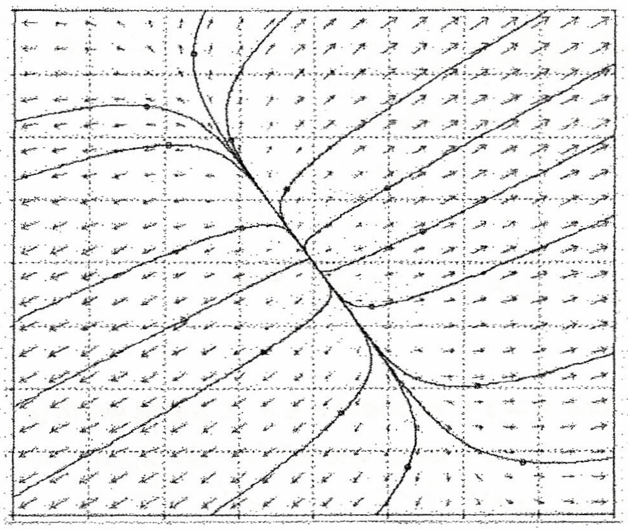

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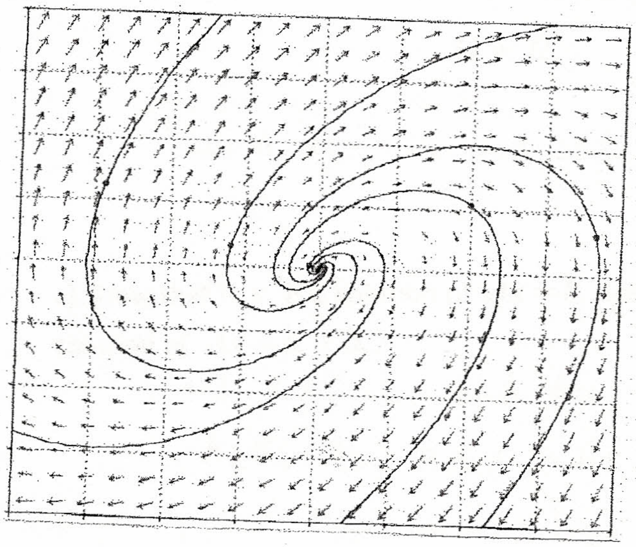

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